DEPARTMENT OF PHYSICS
IMPERIAL COLLEGE LONDON

MSC DISSERTATION

An Introduction to Loop Quantum Gravity with Application to Cosmology

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September 2014

Submitted in partial fulfilment of the requirements for the degree of Master of Science of Imperial College London

Abstract

The development of a quantum theory of gravity has been ongoing in the theoretical physics community for about 80 years, yet it remains unsolved. In this dissertation, we review the loop quantum gravity approach and its application to cosmology, better known as loop quantum cosmology. In particular, we present the background formalism of the full theory together with its main result, namely the discreteness of space on the Planck scale. For its application to cosmology, we focus on the homogeneous isotropic universe with free massless scalar field. We present the kinematical structure and the features it shares with the full theory. Also, we review the way in which classical Big Bang singularity is avoided in this model. Specifically, the spectrum of the operator corresponding to the classical inverse scale factor is bounded from above, the quantum evolution is governed by a difference rather than a differential equation and the Big Bang is replaced by a Big Bounce.

Acknowledgement

In the name of Allah, the Most Gracious, the Most Merciful.

All praise be to Allah for giving me the opportunity to pursue my study of the fundamentals of nature. In particular, I am very grateful for the opportunity to explore loop quantum gravity and its application to cosmology for my MSc dissertation. I would like to express my utmost gratitude to Professor João Magueijo for his willingness to supervise me on this endeavour. I am indebted to him for his positive and encouraging comments throughout my efforts to complete this dissertation. Also, I am thankful to Scott from Cambridge Proofreading LTD for reading the draft, pointing out typos and suggesting better sentences.

I would also like to convey my deepest appreciation to my mother, Wan Ashnah Wan Hussin, and to my siblings, especially Wan Mohd Fakri Wan Mokhtar and Wan Ruwaida Wan Mokhtar. With their endless support and encouragement, I have been able to achieve so much more than possibly thought. In addition, I would like to thank the Indigenous People's Trust Council (MARA) of Malaysia for financially supporting my study here at Imperial College London.

Finally, I would like to extend my appreciation to all lecturers of the MSc in Quantum Fields and Fundamental Forces (QFFF) programme and to all my friends who have been very helpful throughout my study here. In particular, I would like to thank my personal advisor, Professor Daniel Waldram, and my friends, Euibyung Park and Nipol Chaemjumrus.

Contents

1 Introduction

Quantum mechanics as a field was established in the late 19$^{\text{th}}$ century when Max Planck made a formal assumption that energy was transferred discretely in order to derive his blackbody radiation law. Then, Einstein used this idea to explain the photoelectric effect, followed by Bohr, who employed the idea of quantised energy to develop his atomic model. It was not long before physicists of the era realised that the concept plays a fundamental role in describing nature. Over time, quantum mechanics was refined and extended. Several quantisation procedures were developed, for instance, Dirac's canonical quantisation [1] and Feynman's path integral formulation [2]. Eventually, these developments led to the U(1) × SU(2) × SU(3) standard model of particle physics.

Around the same time as the quantum revolution began, there was another revolution taking place. Einstein's theory of special relativity, which was later generalised to his theory of general relativity, transformed how space and time were viewed. Space and time were united into a single entity known as spacetime, and gravitational effects were incorporated into the curvature of spacetime. More importantly, spacetime itself was no longer a fixed background in which physics takes place. Instead, it was promoted to a dynamical entity influenced by the distribution of matter in it. The relationship between spacetime and matter is governed by the Einstein equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}$$

where $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor that encodes some information about the curvature of spacetime given by the metric $g_{\mu\nu}$, R is the Ricci scalar and $T_{\mu\nu}$ is the stress-energy tensor that encodes how matter is distributed in spacetime.

General relativity, however, is a purely classical theory. It does not incorporate any idea of quantum mechanics into the formulation. As early as 1916, which was only one year after Einstein finalised the theory, he pointed out that quantum effects must lead to modifications in general relativity [3]. The development of a quantum theory of gravity (or quantum qravity for short) began. Despite various efforts and ideas being put forward, there has been no single complete theory of quantum gravity until today -

99 years after general relativity was completed.

There are several reasons why the development of a suitable theory has been very difficult. Firstly, the effects of quantum gravity are expected to be significant on the Planck scales:

$$\begin{aligned} \text{Planck energy}, E_P &\equiv \sqrt{\frac{\hbar c^5}{G}} \approx 1.22 \times 10^{19} \text{ GeV} \\ \text{Planck length}, l_P &\equiv \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \times 10^{-33} \text{ cm} \\ \text{Planck time}, t_P &\equiv \sqrt{\frac{\hbar G}{c^5}} \approx 5.39 \times 10^{-44} \text{ s} \end{aligned}$$

which are very remote from everyday life. In fact, even the Large Hadron Collider (LHC), which collides beams of protons on an energy scale of TeV, is nowhere near the regime of interest. As such, we have neither direct observational nor experimental results to guide us in our pursuit of a theory for quantum gravity. This also means that it will not be easy to test the prediction of quantum gravity theory.

Secondly, quantum mechanics and general relativity are counter-intuitive, or at least the ideas are not easily digested. Thus, there is a high probability that the combination of the two will not be intuitive either. Physicists, then, have to turn their attention to what they believe is true or important among the ideas in current theories and work for a mathematically consistent theory of quantum gravity.

Thirdly, there are apparent and genuine conceptual incompatibilities and technical difficulties between quantum mechanics and general relativity. For example, quantum mechanics is probabilistic in nature, while general relativity is deterministic. In addition, quantum mechanics is usually formulated in the presence of a background metric, while in general relativity the metric itself is dynamic.

Having said that quantum gravity effects are very remote from everyday life, why then do we bother constructing a theory of quantum gravity? Firstly, we note that the Einstein equations have the stress-energy tensor of matter on the right-hand side. In this expression, $T_{\mu\nu}$ is purely classical. The standard model, on the other hand, tells us that matters in the universe are best described by quantum mechanics. So, the Einstein equation must somehow be modified to accommodate this fact. The modification may eventually lead to non-trivial and surprising consequences.

Secondly, the theory of general relativity on its own is incomplete. The presence of singularity in general relativity means that the theory breaks down somewhere near the singularity. With respect to Big Bang singularity, for instance, the energy density of matter diverges. Before reaching this point, the universe is very small and dense, implying that both quantum mechanics and general relativity are required to describe the situation.

Loop Quantum Gravity

Loop Quantum Gravity [4, 5, 6, 7, 8, 9, 10] is one of the approaches to achieve the goal. It originates from the attempt to quantise general relativity using canonical methods. As a result of starting from general relativity and taking the lessons from it seriously [10], one is led to a mathematically rigorous, non-perturbative background independent theory of quantum gravity.

In loop quantum gravity, a basis state $|s\rangle$ describing space is represented by a collection of connected curves known as a knotted spin network (s-knot) state. An example of such a state is shown in Figure 1. This will be reviewed in more detail in Section 3 later. The curves are initially defined to be embedded in a Riemannian manifold Σ. However, due to the diffeomorphism invariance inherited from general relativity, the positions of the curves on the manifold lose their meanings. Important information is encoded in the combinatorics between the structures associated with the curves and their meeting points.

Even though it is not yet complete, the theory already gives some important insights on the nature of space on the Planck scale. Specifically, it predicts that space is indeed granular on such a scale. This prediction follows from the discovery that the operators corresponding to classical area and volume have discrete spectra [12, 13, 14]. By the standard interpretation of quantum mechanics, this suggests that the physical measurement of area and volume will give quantised results.

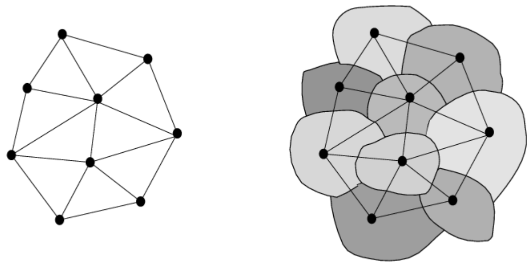

Figure 1: (Left) An example of an s-knot state, $|s\rangle$. The curves are referred to as links, while their meeting points, represented as dots in the diagram, are referred to as nodes. (Right) The interpretation of $|s\rangle$ as quanta of space it describes. This figure is taken from [11].

The spectrum of the area operator depends only on the SU(2)-spin associated with the links in $|s\rangle$, while that of the volume operator depends only on the "intertwiner" associated with the nodes. These results offer a compelling physical interpretation of the s-knot state. It can be viewed as a set of quanta of space (represented by the nodes), each with a particular size, connected by the surface (represented by the links) [7]. This interpretation is illustrated in Figure 1.

In Section 3, we will review the kinematical structures of the theory. Then, these important results will be made more precise. In particular, the spectra of area and volume operators will be presented. Then, we will briefly review the implementation of an important operator, namely the Hamiltonian constraint, which is necessary to obtain a spacetime picture of loop quantum gravity instead of just space. This part of the theory, however, is one of the main open problems yet to be solved [7]. As such, the physical interpretation is not very clear.

Loop Quantum Cosmology

An important area of research closely related to loop quantum gravity is loop quantum cosmology [15, 16, 17, 18, 19]. In this line of research, the methods from loop quantum gravity are applied to a symmetry-reduced setting, specifically to cosmology. In this way, one is able to avoid some technical difficulties inherent to the full theory. Also, the setting is well-suited to address the deep conceptual issues in quantum gravity such as the problem of time and extraction of dynamics from a theory that has no "time evolution". More importantly, it opens up the possibility of addressing quantum gravity theories with observations .

The main result in loop quantum cosmology is the resolution of the classical Big Bang singularity in some simple models of cosmology [20, 21, 22, 23]. There are two aspects in the resolution: (i) The operator corresponding to the inverse scale factor in the classical limit is bounded from above. This resolves the divergent quantities at the Big Bang. (ii) The evolution of the universe does not suddenly "cut off" at the Big Bang. One can continue to model the evolution of the universe beyond the classical singularity point, leading to a pre-Big-Bang universe.

We shall begin by reviewing some aspects of general relativity that are important to the discovery and development of loop quantum gravity in Section 2. In particular, the Hamiltonian formulation will be presented, followed by the introduction of the Ashtekar-Barbero variable, holonomy and flux. After reviewing loop quantum gravity in Section 3, we will proceed to review loop quantum cosmology in Section 4. Finally, we will conclude with some remarks on aspects of both lines of research that are not discussed here, including some important open problems.

2 General Relativity

In general relativity, spacetime is described by a metric field $g(x)$ on a background manifold $\mathcal{M}$. The dynamics of $g(x)$ is encoded in the Einstein-Hilbert action

$$S_{EH}[g] = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-\det g}R[\Gamma(g)] \tag{2.1}$$

where R is the Ricci scalar and $\Gamma(g)$ is the Christoffel symbol. The variation of this action with respect to the metric results in the Einstein equations in vacuum ($T_{\mu\nu} = 0$). One can add matter to the system simply by adding the relevant action to (2.1).

Although the phrase "dynamics" is used, it is not in the usual sense of field evolution through time. Instead, it is about the determination of the field for the whole of spacetime. This should not come as a surprise since general relativity is a theory about spacetime. So, it makes no sense to speak of "spacetime evolution through time". For example, consider the Schwarzschild metric and Friedmann-Lemaitre-Robertson-Walker (FLRW) metric. Each describes a universe, and there is no scenario in which one will evolve into the other at a later time. This feature of general relativity persists even in the Hamiltonian formulation and quantum gravity, leading to what is known as the problem of time.

2.1 Hamiltonian Formulation

In order to quantise a theory using canonical methods, one has to rewrite it in the Hamiltonian form first. This subsection will review such a formulation of general relativity based on [5, 24, 25]. This version of general relativity is also known as the ADM formulation in honour of Arnowitt, Deser and Misner [26].

To put a theory into its Hamiltonian form, we have to identify the appropriate configuration variables and define the corresponding conjugate momenta, which are related to the temporal derivative of the configuration variables. Thus, we have to identify a "time" parameter in the theory. In the case of general relativity, this procedure departs from manifest covariance of the theory.

This requirement is automatically satisfied if we set spacetime $(\mathcal{M}, g_{\mu\nu})$ to be globally hyperbolic. In such spacetime, we can define a global time function t and foliates it into a family of Cauchy hypersurfaces Σ labelled by t [27, 28]. We can then describe general relativity in terms of spatial metric h_{ab} on Σ evolving through time $t \in \mathbb{R}$. However, one must be aware that establishing this condition restricts the spacetime to have the topology $\mathcal{M} \cong \mathbb{R} \times \Sigma$ whereas general relativity in its original formulation can have arbitrary topology [4]. Also, note that t is not necessarily a physical time.

With the 3+1 decomposition above, a general metric can be written as

$$\mathrm{d}s^2 = -N^2 \mathrm{d}t^2 + h_{ab} \left(\mathrm{d}x^a + N^a \mathrm{d}t\right) \left(\mathrm{d}x^b + N^b \mathrm{d}t\right) \tag{2.2}$$

where N and N^a are the lapse function and shift vector, respectively. The information about the spacetime metric $g_{\mu\nu}$ is now completely encoded in h_{ab}, N and N^a. Given a Cauchy hypersurface and the associated Riemannian metric, its relation to the neighbouring hypersurface is given by the extrinsic curvature

$$K_{ab} = \frac{1}{2N} \left(\dot{h}_{ab} - \nabla_a N_b - \nabla_b N_a\right) \tag{2.3}$$

where the dot indicates derivative with respect to t and ∇_c is the covariant derivative on the hypersurface, compatible with h_{ab}.

Written in terms of the structures on Σ, the Einstein-Hilbert action (2.1) takes the form

$$S = \frac{1}{16\pi G} \int \mathrm{d}^4 x N \sqrt{\det h} \left({}^{(3)}R - K_{ab} K^{ab} - \left(K_a{}^a\right)^2\right) \tag{2.4}$$

where ${}^{(3)}R$ is the Ricci scalar on the hypersurface. It depends only on the spatial metric h_{ab} and its spatial derivative. By analysing the action, we can derive the canonical momentum conjugate to h_{ab},

$$\begin{aligned} p^{ab}(\mathbf{x}) &\equiv \frac{\delta L}{\delta \dot{h}_{ab}(\mathbf{x})} = \frac{1}{2N}\frac{\delta L}{\delta K_{ab}(\mathbf{x})} \\ &= \frac{\sqrt{\det h}}{16\pi G}\left(K_{ab} - K^c_c h^{ab}\right). \end{aligned} \tag{2.5}$$

These variables satisfy the Poisson bracket

$$\{h_{ab}(\mathbf{x}), p^{cd}(\mathbf{y})\} = \delta^c_{(a}\delta^d_{b)}\delta(\mathbf{x},\mathbf{y}). \tag{2.6}$$

The canonical momenta conjugate to N and N^a, on the other hand, are zero since no time derivative of these variables appear in the action. These types of variables are known as Lagrange multipliers and are not actually configuration variables. So, we identify that in the Hamiltonian formulation of general relativity, the only configuration variable is the spatial metric, h_{ab}. The phase space variables are then h_{ab} and p^{cd} satisfying Poisson bracket (2.6).

Although the Lagrange multipliers do not serve as configuration variables, they do have an important role. Setting the action to be invariant under arbitrary variation of the Lagrange multipliers yields a set of equations of the form $C_i = 0$, where C_i are known as the constraints of the system. These constraint equations are meant to be applied after the Poisson bracket structure of the canonical variables is constructed. They are sometimes written with a symbol $\approx$ rather than $=$ to indicate this feature. For any system with constraints, not all points on the phase space are physically relevant. Instead, only those that satisfy all the constraint equations are physically relevant.

For general relativity, the variation with respect to N gives the Hamiltonian or scalar constraint

$$C^{grav} = \frac{16\pi G}{\sqrt{\det h}}\left(p_{ab}p^{ab} - \frac{1}{2}(p^c_c)^2\right) - \frac{N\sqrt{\det h}}{16\pi G}R \approx 0 \tag{2.7}$$

while variation with respect to N^a gives the diffeomorphism or vector constraint

$$C^{grav}_a = -2D_b p^b_a \approx 0. \tag{2.8}$$

Other than reducing the number of physically relevant phase space points, these constraints also encode the symmetry of the system. The diffeomorphism constraint, for example, generates spatial diffeomorphism of the phase space variables. The action of the Hamiltonian constraint, on the other hand, is more complicated. This is discussed in more detail in [5, 24, 25].

Given an action of a system, one will now proceed to calculate the Hamiltonian that would generate the time evolution for the system. In our case now, the Hamiltonian obtained from the action (2.4) is

$$\begin{aligned} H_{grav} &= \int \mathrm{d}^3x \left(\dot{h}_{ab} p^{ab} - L_{grav} \right) \\ &= \int \mathrm{d}^3x \left(\frac{16\pi G}{\sqrt{\det h}} \left(p_{ab} p^{ab} - \frac{1}{2} (p^c_c)^2 \right) + 2 p^{ab} D_a N_b - \frac{N\sqrt{\det h}}{16\pi G} R \right) \\ &= \int \mathrm{d}^3x \left(N C^{grav} + N^a C^{grav}_a \right) \end{aligned} \tag{2.9}$$

where we have to integrate by parts to arrive at the third line. One should immediately recognise that for physically relevant situations, the Hamiltonian vanishes since it consists entirely of constraints. This gives rise to the problem mentioned at the beginning of this section, namely the problem of time. With the vanishing Hamiltonian, one is led to a theory with apparently no “time evolution”.

2.2 Ashtekar-Barbero Variable

In 1986, Ashtekar introduced a new complex variable that puts general relativity in the language of gauge theory [29]. The real version was suggested later by Barbero [30] in 1995. The introduction of the variable was a crucial step towards the development of loop quantum gravity. It allows one to employ, or at least be guided by, methods in gauge theory, which was better understood for the purpose of quantisation. This subsection, based on [16, 24, 25, 31], will review the Hamiltonian formulation of general relativity in terms of the new variable.

To introduce the Ashtekar-Barbero variable, we first consider a set of three vectors $e_{(i)} = e^a_i \partial_a$, where $i = 1, 2, 3$, at each point in space. These vectors are taken to be orthonormal to each other; that is:

$$h_{ab}e^a_i e^b_j = \delta_{ij}. \tag{2.10}$$

We can also define a set of three co-vectors $e^{(i)} = e^i_a \mathrm{d}x^a$, where $i = 1, 2, 3$, and demand that $e^{(i)}(e_{(j)}) = \delta^i_j$. With this condition, e^i_a and e^a_i become uniquely inverse to each other. We can then invert (2.10) to obtain

$$h_{ab} = \delta_{ij} e^i_a e^j_b, \tag{2.11}$$

from which we can see that specifying e^i_a (or e^a_i) is equivalent to specifying h_{ab}. However, the use of triads introduces an internal SO(3) symmetry to the theory since $R^i{}_j e^j_a$ where $R^i{}_j \in$ SO(3) gives the same h_{ab} as e^j_a. One can also view the triads and co-triads as $\mathfrak{su}(2)$-valued vectors and co-vectors, respectively. Under an SU(2) gauge transformation, the internal indices $i, j, k, ...$ transform in the vector representation. In this way, the SO(3) symmetry is replaced by SU(2) symmetry.

The variable of interest is not actually the triads themselves. Instead, it is the densitised triads

$$E^a_i \equiv \sqrt{\det h}\; e^a_i = |\det(e)| e^a_i \tag{2.12}$$

which will later serve as the canonical momentum. This introduces additional symmetry to the theory. Due to the absolute value of the determinant of e^j_b in (2.12), the theory is invariant under orientation (left-handed or right-handed) of the triads.

Another structure that we need to introduce is the spin connection $\omega_a{}^i{}_j$ that appears in the definition of SU(2) gauge covariant derivative:

$$D_a v^i = \partial_a v^i + \omega_a{}^i{}_j v^j \tag{2.13}$$

where v^i is an arbitrary $\mathfrak{su}(2)$-valued function. By defining $\Gamma^i_a \equiv \frac{1}{2}\omega_{ajk}\varepsilon^{ijk}$, the Ashtekar-Barbero connection is then defined as

$$A^i_a \equiv \Gamma^i_a + \gamma K^i_a \tag{2.14}$$

where $K_a^i \equiv \delta^{ij} K_{ab} e_j^b$ is the extrinsic curvature in mixed indices and $\gamma > 0$ is the Barbero-Immirzi parameter [30, 32]. A_a^i is an $\mathfrak{su}(2)$-valued one-form transforming as a connection under gauge transformation, that is, $A_a \to g(A_a + \partial_a)g^{-1}$ where $g \in$ SU(2). The important fact is that A_a^i and E_j^b are conjugate variables satisfying Poisson bracket

$$\{A_a^i(\mathbf{x}), E_j^b(\mathbf{y})\} = 8\pi\gamma G \delta_j^i \delta_a^b \delta(\mathbf{x}, \mathbf{y}). \tag{2.15}$$

We can fully describe general relativity using these new variables together with the constraints (2.7) and (2.8) in the appropriate form. The Hamiltonian constraint now takes the form

$$C^{grav} = \left(\epsilon^{ijk} F_{ab}^i - 2(1+\gamma^2)(A_a^i - \Gamma_a^i)(A_b^j - \Gamma_b^j)\right) \frac{E_j^{[a} E_k^{b]}}{\sqrt{|\mathrm{det} E|}} \approx 0 \tag{2.16}$$

where

$$F_{ab}^i = \partial_a A_b^i - \partial_b A_a^i - \epsilon^{ijk} A_a^j A_b^k \tag{2.17}$$

is the Yang-Mills curvature and the diffeomorphism constraint reads

$$C_a^{grav} = F_{ab}^i E_i^b \approx 0 \tag{2.18}$$

In addition, a new constraint arises from the use of triads. The Gauss constraint

$$\begin{aligned} D_a^{(A)} E_i^a &\equiv \partial_a E_i^a + \epsilon^{ijk} A_a^j E_k^a \\ &= D_a E_i^a + \epsilon^{ijk} K_a^j E_k^a \\ &\approx 0 \end{aligned} \tag{2.19}$$

generates SU(2) gauge symmetry in the theory.

2.3 Holonomy and Flux

Before we end this section on general relativity, we would like to introduce another set of variables that will actually be promoted to basic operators in loop quantum gravity. These variables, namely holonomy and flux, smear A_a^i and E_j^b fields, respectively, in a background independent way. As a result, one is able to obtain well-defined Poisson brackets without the Dirac delta function.

Given a manifold Σ and an oriented curve $e \in \Sigma$, a holonomy h_e is defined as

$$h_e[A] = \mathcal{P}\text{exp}\left(G \int_e \mathrm{d}\lambda \dot{e}^a A_a^i \tau_i \right) \tag{2.20}$$

where G is Newton's gravitational constant, $\tau_i \equiv -\frac{i}{2}\sigma_i$, with $i = 1, 2, 3$, is a basis of $\mathfrak{su}(2)$ and σ_i are the Pauli matrices. $\dot{e}^a$ is the tangent vector to curve e parametrised by λ. The symbol $\mathcal{P}$ is to denote that the integration should be carried out in a path-ordered manner. Note that the holonomy is coordinate-independent, but is not gauge-invariant. Under a gauge transformation, the holonomy transforms as

$$h_e[A] \rightarrow g_{s(e)} h_e[A] g_{t(e)}^{-1} \tag{2.21}$$

where $s(e)$ denotes the source or starting point of e and $t(e)$ denotes the target or ending point.

Given a surface $S \in \Sigma$ with local coordinates y^a, a flux F_f^S is defined as

$$F_f^S[E] = \int_S \mathrm{d}^2 y n_a E_i^a f^i \tag{2.22}$$

where f^i is an $\mathfrak{su}(2)$-valued function, $n_a = \frac{1}{2}\varepsilon_{abc}\varepsilon^{uv}\frac{\partial x^b}{\partial y^u}\frac{\partial x^c}{\partial y^v}$ is the co-normal to the surface S and x^a is the local coordinate of Σ. From the expression, it is obvious that the flux is both coordinate-independent and gauge-invariant.

3 Loop Quantum Gravity

Historically, the first attempt to canonically quantise general relativity was made using the spatial metric h_{ab} and its conjugate momentum p^{ab} as the basic variables. Following Dirac's procedure [1], these variables were promoted to operators on a kinematical Hilbert space $\mathcal{H}_{kin}$

$$h_{ab} \to \hat{h}_{ab} \qquad p^{ab} \to \hat{p}^{ab} \tag{3.1}$$

such that the Poisson bracket (2.6) between them is promoted to a commutation relation

$$[\hat{h}_{ab}(\mathbf{x}), \hat{p}^{cd}(\mathbf{y})] = i\hbar\delta^c_{(a}\delta^d_{b)}\delta(\mathbf{x},\mathbf{y}). \tag{3.2}$$

Then, one chooses a representation space to study the action of the operators (3.1) on a general quantum state $|\Psi\rangle \in \mathcal{H}_{kin}$. For instance, in metric representation, we would have

$$\hat{h}_{ab}(\mathbf{x})\Psi[h_{ab}(\mathbf{x})] = h_{ab}(\mathbf{x})\Psi[h_{ab}(\mathbf{x})] \tag{3.3}$$

$$\hat{p}^{cd}(\mathbf{x})\Psi[h_{ab}(\mathbf{x})] = -i\hbar\frac{\delta}{\delta h_{cd}(\mathbf{x})}\Psi[h_{ab}(\mathbf{x})]. \tag{3.4}$$

Among the elements of $\mathcal{H}_{kin}$, only those that are annihilated by both quantum versions of constraints (2.7) and (2.8)

$$\hat{C}^{grav}\Psi[h_{ab}(\mathbf{x})] = 0 \qquad \hat{C}^{grav}_a\Psi[h_{ab}(\mathbf{x})] = 0 \tag{3.5}$$

are physically relevant. They are elements of the physical Hilbert space $\mathcal{H}_{phys}$. One can also implement the diffeomorphism constraint (2.8) alone first to identify a diffeomorphism-invariant Hilbert space $\mathcal{H}_{diff}$. In this way, one will have a chain of Hilbert space construction

$$\mathcal{H}_{kin} \xrightarrow{\hat{C}^{grav}_a} \mathcal{H}_{diff} \xrightarrow{\hat{C}^{grav}} \mathcal{H}_{phys}. \tag{3.6}$$

However, note that constraints are not necessarily implemented as operators annihilating wave functionals as described in (3.5). Sometimes, it is more convenient to identify the restrictions implied by the constraints and implement them, for example, via a group-averaging procedure.

Loop quantum gravity, more or less, follows a similar path. Instead of working the metric variables (h_{ab}, p^{cd}), we will work with the connection variables (A^i_a, E^b_j). As we have mentioned above, using these variables introduces another constraint (2.19). Therefore, there will be an additional chain in (3.6) from $\mathcal{H}_{kin}$ to $\mathcal{H}^{inv}_{kin}$ before we can arrive at $\mathcal{H}_{diff}$. We start by introducing the kinematical Hilbert space $\mathcal{H}_{kin}$ following [5].

3.1 Cylindrical Functions

The kinematical Hilbert space of loop quantum gravity is related to the concept of holonomy introduced in Section 2.3 and uses the notion of cylindrical functions. Instead of just a single curve e as in the definition of holonomy above, consider a graph Γ defined as a collection of oriented paths $e \in \Sigma$ meeting at most at their endpoints (see Figure 2 for an example). The paths are usually referred to as links or edges in loop quantum gravity literature.

Given a graph $\Gamma \in \Sigma$ with L links, one can associate a smooth function $f : \mathrm{SU}(2)^L \to \mathbb{C}$ with it. A cylindrical function is a couple (Γ, f), which in connection representation is defined as a functional of A given by

$$\langle A|\Gamma, f\rangle = \psi_{(\Gamma,f)}[A] = f(h_{e_1}[A], ..., h_{e_L}[A]) \tag{3.7}$$

where e_l with $l = 1, ..., L$ are the links of the corresponding graph Γ. The space of all functions f associated with a particular graph Γ is denoted as Cyl_Γ.

Since the holonomies are simply an element of SU(2), a scalar product between elements of Cyl_Γ can be defined as

$$\langle \Gamma, f|\Gamma, g\rangle \equiv \int \mathrm{d}h_{e_1}...\mathrm{d}h_{e_L}\overline{f(h_{e_1}[A], ..., h_{e_L}[A])}g(h_{e_1}[A], ..., h_{e_L}[A]) \tag{3.8}$$

where $\mathrm{d}h_{e_n}$ is the Haar measure. This turns Cyl_Γ into a Hilbert space $\mathcal{H}_\Gamma$ associated

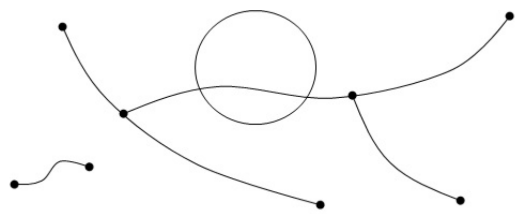

Figure 2: An example of a graph Γ with seven curves. The dots are the endpoints of the curves.

with a given graph Γ.

The kinematical Hilbert space $\mathcal{H}_{kin}$ is then defined as $\mathcal{H}_{kin} \equiv \bigoplus_{\Gamma\subset\Sigma} \mathcal{H}_\Gamma$ with the scalar product

$$\begin{aligned} \langle\Gamma_1, f_1|\Gamma_2, f_2\rangle &\equiv \langle\Gamma_1\cup\Gamma_2, f_1|\Gamma_1\cup\Gamma_2, f_2\rangle \\ &\equiv \int \mathrm{d}\mu_{AL}\overline{\psi_{\Gamma_1,f_1}[A]}\psi_{\Gamma_2,f_2}[A] \end{aligned} \tag{3.9}$$

where $\mathrm{d}\mu_{AL}$ is Ashtekar-Lewandowski measure [33, 34, 35].

3.2 Loop States

An example of cylindrical function is the case where each e is a single closed curve (that is, a loop) α and f is the trace function, Tr [10]. For a graph with a collection of loops, that is, $\Gamma \equiv \alpha = (\alpha_1, ..., \alpha_n)$, we have[1]

$$\langle A|\alpha\rangle = \psi_\alpha[A] = \psi_{\alpha_1}[A]...\psi_{\alpha_n}[A] = \text{Tr } h_{\alpha_1}[A]...\text{Tr } h_{\alpha_n}[A] \tag{3.10}$$

where Tr $h_\alpha[A] =$ Tr $\mathcal{P}$exp $\oint_\alpha A$ is known as the Wilson loop. These functions are called (multi-)loop states and play an important role in the history of loop quantum gravity. They are the reason why loop quantum gravity bears the word "loop".

Historically, they were found to be exact solutions to the quantum version of the Gauss constraint (2.19) and Hamiltonian constraint (2.16) [36]. This suggested that one

[1]For these types of functions, we suppress the $f \equiv$ Tr label.

can expand a general quantum state $\Psi[A]$ in terms of these multi-loop states

$$\langle A|\Psi\rangle \equiv \Psi[A] = \sum_{\alpha} \Psi(\alpha)\psi_{\alpha}[A] \tag{3.11}$$

where $\Psi(\alpha)$ is the loop space representation of the state $|\Psi\rangle$ [37]. In fact, using (3.9) one can ascertain, for instance, if α consists of only a single loop

$$\langle \alpha|\Psi\rangle \equiv \Psi(\alpha) = \int \mathrm{d}\mu_{AL}\ \mathrm{Tr}\ \mathcal{P}\mathrm{e}^{\oint_{\alpha} A}\Psi[A] \tag{3.12}$$

which bears close resemblance to momentum space representation in ordinary quantum mechanics. The transformation (3.12) is known as "loop transform".

Despite being very useful, these multi-loop states form an overcomplete basis and complicates the formalism [10]. This overcompleteness problem was solved when the spin network states (to be described in the next section) were found to be a genuine gauge-invariant orthonormal basis [38].

3.3 Spin Network States

To introduce spin network states, we start by introducing an orthonormal basis obtained using the Peter-Weyl theorem. The theorem states that a basis of the Hilbert space $\mathcal{H}_{\mathrm{SU(2)}} = L_2(\mathrm{SU}(2), \mathrm{d}\mu_{Haar})$ of functions on SU(2) is given by the matrix elements of the irreducible representations of the group [10]. For example,

$$f(g) = \sum_{j} f^{j}_{mn} D^{(j)}_{mn}(g) \tag{3.13}$$

where $D^{(j)}_{mn}(g)$ is the Wigner matrices that give the spin-j irreducible representation of the group element $g \in$ SU(2). Since $\mathcal{H}_{\Gamma}$ is a tensor product of $\mathcal{H}_{\mathrm{SU(2)}}$, the basic elements are

$$\begin{aligned} \langle A|\Gamma, j_L, m_L, n_L\rangle &\equiv \langle A|\Gamma, j_{e_1}, ..., j_{e_L}, m_{e_1}, ..., m_{e_L}, n_{e_1}, ..., n_{e_L}\rangle \\ &= D^{(j_1)}_{m_1 n_1}(h_{e_1}[A])...D^{(j_L)}_{m_L n_L}(h_{e_L}[A]). \end{aligned} \tag{3.14}$$

These constructions can easily be extended to $\mathcal{H}_{kin}$ by labelling the basis not only with the j's, m's and n's but also Γ. This basis is obviously not gauge-invariant. Under gauge transformation (2.21), each Wigner matrix also transforms at its endpoints:

$$D^{(j)}(h_e) \to D^{(j)}(g_{s(e)} h_e g^{-1}_{t(e)}) = D^{(j)}(g_{s(e)}) D^{(j)}(h_e) D^{(j)}(g^{-1}_{t(e)}). \tag{3.15}$$

Let us now consider a graph Γ with L links such that the endpoints of each curve necessarily meet another endpoint. These meeting points are known as nodes or vertices. Let N denote the number of nodes in Γ. Then, the gauge-invariant basis can be obtained by group averaging

$$\begin{aligned}&[D^{(j_1)}_{m_1 n_1}(h_{e_1})...D^{(j_L)}_{m_L n_L}(h_{e_L})]_{inv} \\ &\qquad \equiv \int \prod_{n=1}^{L} \mathrm{d}g_n D^{(j_1)}_{m_1 n_1}(g_{s(e_1)} h_{e_1} g^{-1}_{t(e_1)})...D^{(j_L)}_{m_L n_L}(g_{s(e_L)} h_{e_L} g^{-1}_{t(e_L)})\end{aligned} \tag{3.16}$$

which eventually amounts to associating each node n with an intertwiner i_n [5]. For a node n where k links meet, an intertwiner associated with it is an element from the invariant subspace of the Hilbert space $\mathcal{H}_n$

$$\mathcal{H}_n = \mathcal{H}_{j_1} \otimes ... \otimes \mathcal{H}_{j_k}. \tag{3.17}$$

This then defines a gauge-invariant spin network state $|S\rangle \equiv |\Gamma, j_1, ..., j_L, i_1, ...i_N\rangle$

$$\langle A|S\rangle = \bigotimes_{l=1}^{L} D^{(j_l)}(h_{e_i}[A]) \cdot \bigotimes_{n=1}^{N} i_n \tag{3.18}$$

where $\cdot$ is to denote that indices of matrix elements of $D^{(j)}$ and i_n contract appropriately and give a gauge-invariant result.

For example, consider a spin network state $|S\rangle$ which is represented by a graph with three links $l_1 = 1, l_2 = 2, l_3 = 3$ and two nodes n_1, n_2 as shown in Figure 3. The links are associated with spins $j_1 = 1, j_2 = 1/2, j_3 = 1/2$ respectively. Then, in connection

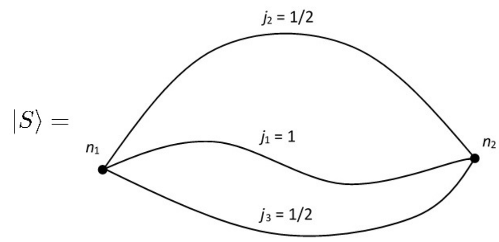

Figure 3: A spin network state $|S\rangle$ with three links l_1, l_2, l_3 labelled by the respective representation j_1, j_2, j_3 of the associated holonomies and two nodes n_1, n_2.

representation, it is given by [10]

$$\langle A|S\rangle = \frac{1}{3}\sigma_{iAB}\sigma^{jCD}(D^{(1)}(h[A]))^i{}_j(D^{(1/2)}(h[A]))^A{}_C(D^{(1/2)}(h[A]))^B{}_D \quad (3.19)$$

where σ_i are the Pauli matrices, which happen to play the role of intertwiners for this particular case.

With a complete gauge-invariant orthonormal [10] basis available, a general quantum state made up of superposition of these spin network states (3.18) will be an element of $\mathcal{H}_{kin}^{inv}$ and solves the Gauss constraint.

3.4 Quanta of Area and Volume

At this stage, we can already present the main results from loop quantum gravity, namely the discreteness of area and volume [12, 13, 14]. For this, we have to know how basic operators act on a general quantum state. However, $A_a^i(\mathbf{x})$ and $E_j^b(\mathbf{x})$ are not well-defined on $\mathcal{H}_{kin}$ because they would send wave functionals $\Psi[A] \in \mathcal{H}_{kin}$ out of the state space [10]. Therefore, holonomies and fluxes, which are well-defined, are taken as the basic operators instead.

A holonomy operator $\hat{h}_e[A]$ simply acts as a multiplicative factor in connection representation:

$$\hat{h}_e[A]\Psi[A] = h_e[A]\Psi[A]. \quad (3.20)$$

The action on a flux operator, on the other hand, is a bit complicated. $E_j^b(\mathbf{x})$ will still be promoted as a functional derivative so that the action of a flux operator is

$$\hat{F}_i^S \Psi[A] = -8\pi\gamma i\hbar \int_S \mathrm{d}^2 y n_a \frac{\delta}{\delta A_a^i(\mathbf{x}(\mathbf{y}))} \Psi[A]. \tag{3.21}$$

For the case of $\Psi[A] = D^{(j)}(h_e[A])$ and e only intersect S at point p, that is, $e \cap S = p$, (3.21) will be [24]

$$\hat{F}_i^S D^{(j)}(h_e[A]) = \pm 8\pi\gamma i\hbar G D^{(j)}(h_{e_1}[A]) \tau_i^{(j)} D^{(j)}(h_{e_2}[A]) \tag{3.22}$$

where $\tau_i^{(j)}$ is the SU(2) generator in spin-j representation, $D^{(j)}(h_{e_1}[A])$ is the spin-j holonomy from $s(e)$ to p and $D^{(j)}(h_{e_2}[A])$ is the spin-j holonomy from p to $t(e)$. Intuitively, the operator $\hat{F}_i^S$ would "cut" the curve at the point of intersection and insert the corresponding spin-j representation generator $\tau_i^{(j)}$ at that point together with the extra factor $\pm 8\pi\gamma i\hbar G$. The sign + or - depends on the relative orientation of the curve and the surface. For multiple intersections, (3.22) is generalised to [10]

$$\hat{F}_i^S D^{(j)}(h_e[A]) = \pm 8\pi\gamma i\hbar G \sum_{p \in (e \cap S)} D^{(j)}(h_{e_1}^p[A]) \tau_i^{(j)} D^{(j)}(h_{e_2}^p[A]). \tag{3.23}$$

However, if there is no intersection, or the curve e is tangential to the surface S, then (3.22) (and thus (3.23)) will vanish.

An important composite operator to consider is the scalar product of two fluxes $\hat{F}_S^2 \equiv \delta^{ij} \hat{F}_i^S \hat{F}_j^S$. The action of this operator follows directly from (3.22). Instead of inserting just a single generator $\tau_i^{(j)}$, this composite operator will insert a sum of them, that is, $\delta^{ij}\tau_i^{(j)}\tau_j^{(j)}$. However, this is equal to $-j(j+1)\mathbb{1}$ which can be commuted through $D^{(j)}(h_{e_1}[A])$ in the expression above and allows the segments e_1 and e_2 of the curve e to reconnect. Therefore, we obtain a simple result:

$$\hat{F}_S^2 D^{(j)}(h_e[A]) = (8\pi\gamma\hbar G)^2 j(j+1) D^{(j)}(h_e[A]) \tag{3.24}$$

which holds when there is only a single intersection between e and S [10].

3.4.1 Area Operator

Classically, the area of a surface $\mathcal{S}$ is given by

$$\begin{aligned} A(\mathcal{S}) &= \int_{\mathcal{S}} \mathrm{d}^2 y \sqrt{\delta^{ij} n_a E^a_i n_b E^b_j} \\ &\approx \lim_{N\to\infty} \sum_{n=1}^{N} \mathcal{S}_n \sqrt{\delta^{ij} n_a E^a_i n_b E^b_j} \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \sqrt{\delta^{ij} F_i^{\mathcal{S}_n} F_j^{\mathcal{S}_n}} \end{aligned} \tag{3.25}$$

where the integral has been written as the limit of a Riemann sum in the second line. The third line follows from the fact that for a small enough surface $\delta\mathcal{S}$, the flux (2.22) can be approximated as

$$F_i^{\delta\mathcal{S}}[E] \approx \delta\mathcal{S} n_a E^a_i. \tag{3.26}$$

The corresponding area operator $\hat{A}(\mathcal{S})$ is then obtained by promoting the flux in (3.25) above into $\hat{F}_i^{\mathcal{S}_n}$ [39]. This definition of area operator ensures that each small surface $\mathcal{S}_n$ intersects at most one time with the curves in a given quantum state. As such, one can immediately use (3.24) to obtain

$$\hat{A}_{\mathcal{S}} |S\rangle = 8\pi\gamma l_P^2 \sum_{p\in(\Gamma\cap\mathcal{S})} j_p(j_p+1) |S\rangle \tag{3.27}$$

which is clearly discrete and finite. Note that the smallest possible area is of order Plank area l_P^2 if the Immirzi parameter γ is taken to be of order unity .

3.4.2 Volume Operator

The definition of the volume operator $\hat{V}_{\mathcal{R}}$ of a region $\mathcal{R}$ follows a similar strategy. One starts by dividing the region into small cells $\mathcal{C}$ of coordinate size ϵ^3 and writing the classical expression for volume as a Riemann sum

$$\begin{aligned} V(\mathcal{R}) &= \int_{\mathcal{R}} \mathrm{d}^3 x \sqrt{\det h} = \int_{\mathcal{R}} \mathrm{d}^3 x \sqrt{|\det E|} \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \epsilon_n^3 \sqrt{\left| \frac{1}{3!} \varepsilon_{abc} \varepsilon^{ijk} E^a_i(\mathbf{x}_n) E^b_j(\mathbf{x}_n) E^c_k(\mathbf{x}_n) \right|} \end{aligned} \tag{3.28}$$

where x_n is an arbitrary point inside the small cell $\mathcal{C}_n$. To express the densitised triads in terms of fluxes, there are two natural ways to proceed [14]. One was introduced by Rovelli and Smolin [12], while the other, by Ashtekar and Lewandowski [34]. Each approach leads to a slightly different volume operator, but maintains the characteristic feature of acting only on the nodes of a graph. More importantly, both operators result in discrete spectra.

In the construction of the Ashtekar-Lewandoski volume operator, for example, one proceeds by choosing three non-coincident surfaces $(\mathcal{S}^1, \mathcal{S}^2, \mathcal{S}^3)$ within each small cell [6]. This allows (3.28) to be written in terms of fluxes, which are then quantised by turning them into operators:

$$\hat{V}(\mathcal{R}) = \lim_{N\to\infty} \sum_{n=1}^{N} \sqrt{\left| \frac{1}{3!} \varepsilon_{abc} \varepsilon^{ijk} \hat{F}_i(\mathcal{S}_n^a) \hat{F}_j(\mathcal{S}_n^b) \hat{F}_k(\mathcal{S}_n^c) \right|}. \tag{3.29}$$

The spectrum of this volume can be obtained using a result similar to (3.23), but is generalised to the action of flux operator on a node. Since this method relies on the choice of three surfaces, one has to carry an averaging procedure over all possible choices of surfaces. This results in a volume operator with the following discrete spectrum [14]

$$\hat{V}(\mathcal{R})\,|S\rangle = (8\pi\gamma l_P^2)^{3/2} \kappa_0 \sum_{n\in S} \sqrt{\left| \frac{1}{48} \varepsilon^{ijk} \sum_{e,e',e''} \epsilon(e,e',e'') \tau_i^{(j_{n,e})} \tau_j^{(j_{n,e'})} \tau_k^{(j_{n,e''})} \right|}\,|S\rangle \tag{3.30}$$

where n are nodes in the spin network state $|S\rangle$ and (e, e', e'') runs over all possible sets of three links at the node n. The factor κ_0 is a constant that came from the averaging procedure, and $\epsilon(e, e', e'')$ is a function associated with the relative orientation of the three links.

3.5 S-Knots

Let us now return to the implementation of the constraints. Since the spin network states $\{|S\rangle\}$ form a basis of $\mathcal{H}_{kin}^{inv}$, it is natural to seek $\mathcal{H}_{diff}$ by analysing the transformation of $|S\rangle$ under a diffeomorphism $\phi \in$ Diff(Σ). The elements of Diff(Σ) can be grouped into three categories according to their action on the spin network states [5, 8]:

(a) Those that leave all labels in $|S\rangle$ invariant. The action of diffeomorphisms in this category is trivial; they would at most shuffle the points inside the links. This category is denoted by $\text{TDiff}_\Gamma(\Sigma)$.

(b) Those that exchange the links in $|S\rangle$ among themselves without changing Γ. This category is obtained by factoring out $\text{TDiff}_\Gamma(\Sigma)$ from the group of all diffeomorphisms that maps Γ to Γ, $\text{Diff}_\Gamma(\Sigma)$. Thus, this category is denoted as $\text{GS}_\Gamma(\Sigma) \equiv \text{Diff}_\Gamma(\Sigma)/\text{TDiff}_\Gamma(\Sigma)$.

(c) Those that move Γ around on the manifold Σ. Analogous to (b), this category is obtained by factoring out $\text{Diff}_\Gamma(\Sigma)$ from $\text{Diff}(\Sigma)$.

Clearly, seeking $\mathcal{H}_{diff}$ amounts to constructing quantum states invariant under any diffeomorphism ϕ that falls into (b) or (c); that is, those in $\text{Diff}(\Sigma)/\text{TDiff}_\Gamma(\Sigma)$.

One way to construct such states is to carry out a group-averaging procedure analogous to (3.16). Given a spin network state $|S\rangle$ associated with a graph Γ, the corresponding invariant state then is defined as [6]

$$|S_{diff}\rangle \equiv \sum_\phi |\phi S\rangle \tag{3.31}$$

where the summation runs over all ϕ in $\text{Diff}(\Sigma)/\text{TDiff}_\Gamma(\Sigma)$ and $|\phi S\rangle$ label the spin network states $|S\rangle$ acted by ϕ. However, there is an important difference between the two group-averaging procedures. While (3.16) results in a gauge-invariant subspace of $\mathcal{H}_{kin}$, (3.31) results in a space outside $\mathcal{H}_{kin}^{inv}$.

The action (of the bra version) of (3.31) on a spin network is defined as [10]

$$\langle S_{diff}|S'\rangle \equiv \sum_\phi \langle \phi S|S'\rangle = \begin{cases} 0 & \text{if there is no } \phi \text{ s.t. } \phi\Gamma = \Gamma' \\ \sum_{\phi_{\Gamma'}} \langle \phi_{\Gamma'} S|S'\rangle & \text{if there exists } \phi \text{ s.t. } \phi\Gamma = \Gamma' \end{cases} \tag{3.32}$$

where the sum of $\phi_{\Gamma'}$ runs over all diffeomorphisms that map Γ to Γ'. The results on the right-hand side follow directly from the fact that the inner product between two spin network states represented by graphs that do not coincide is zero. The inner product between two diffeomorphism-invariant spin network states is then defined as

$$\langle S_{diff} | S'_{diff} \rangle \equiv \langle S_{diff} | S' \rangle \, . \tag{3.33}$$

The definition of inner product (3.33) and the results (3.32) above tell us that $|S_{diff}\rangle$ and $|S'_{diff}\rangle$ are orthogonal unless they belong to an equivalence class K of graphs Γ under diffeomorphisms. The class K is known as knot class. Therefore, instead of being labelled by Γ, the basis states in $\mathcal{H}_{diff}$ is firstly labelled by K [10]. To obtain an orthonormal basis, an additional label c is added to label the colouring of links and nodes in the state. This, then, defines a knotted spin network (s-knot) basis, $\{|s\rangle \equiv |K, c\rangle\}$ for $\mathcal{H}_{diff}$.

3.6 The Problem of Dynamics

The final step before one can really study the true physical consequences of loop quantum gravity is the implementation of the Hamiltonian constraint (2.16). As mentioned in the introduction, this part of the theory is still a work in progress. So, the best way to implement it is not very clear. To implement the Hamiltonian constraint (2.16) as an operator, one usually quantises it following Thiemann's trick [40, 41] and studies its action on s-knot states (or spin network states since the former are constructed from the latter).

Following [41], instead of quantising (2.16) directly, it is more convenient to smear it with the lapse function N first:

$$\begin{aligned} C^{grav}[N] &\propto \int \mathrm{d}^3x N \left(\left(\varepsilon^{ijk} \delta_{il} F^l_{bc} \frac{E^b_j E^c_k}{\sqrt{|\mathrm{det}E|}} \right) - \frac{2(1+\gamma^2)}{\gamma^2} \left(K^j_b K^k_c \frac{E^{[b}_j E^{c]}_k}{\sqrt{|\mathrm{det}E|}} \right) \right) \\ &\equiv C^E[N] - \frac{2(1+\gamma^2)}{\gamma^2} T[N] \end{aligned} \tag{3.34}$$

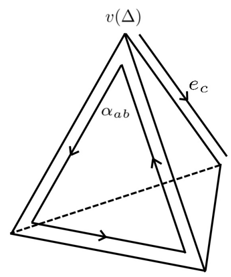

Figure 4: An elementary tetrahedron Δ. This figure is reproduced from [24].

where we have used (2.14) in the second term. Using the identities

$$\varepsilon^{ijk}\frac{E^b_j E^c_k}{\sqrt{|\det E|}} \propto \varepsilon^{abc}\left\{A^i_a, V\right\} \tag{3.35}$$

$$K^i_a \propto \left\{A^i_a, \left\{C^E[1], V\right\}\right\} \tag{3.36}$$

we can put both $C^E[N]$ and $T[N]$ in terms of Poisson brackets

$$C^E[N] \propto \int \mathrm{d}^3x N \varepsilon^{abc}\delta_{ij}F^i_{ab}\left\{A^j_c, V\right\} \tag{3.37}$$

$$T[N] \propto \int \mathrm{d}^3x N \varepsilon^{abc}\varepsilon_{ijk}\left\{A^i_a, \left\{C^E[1], V\right\}\right\}\left\{A^j_b, \left\{C^E[1], V\right\}\right\}\left\{A^k_c, V\right\}. \tag{3.38}$$

Then, similar to the area and volume operator in Section 3.4, we want to write the integral as a Riemann sum. In particular, consider dividing the integration region into a collection of elementary tetrahedra Δ, as illustrated in Figure 4, with edges of coordinate length ϵ (thus the size of each tetrahedron is of order ϵ^3) and choose one vertex $v(\Delta)$. In this way, the holonomy along a loop and an edge can be written as

$$h_{\alpha_{ab}} = 1 + \frac{1}{2}\epsilon^2 F^i_{ab}\tau_i + O(\epsilon^4) \quad \text{and} \quad h_{e_c} = 1 + \epsilon A^i_c \tau_i + O(\epsilon^2) \tag{3.39}$$

respectively, thus allowing the connection and curvature in (3.37) to be written as holonomies. As a result, $C^E[N]$ and $T[N]$ take the form

$$C^E[N] \propto \lim_{\Delta \to v(\Delta)} \sum_{\Delta \in \Sigma} N(v(\Delta))\varepsilon^{abc}\delta_{ij}(h_{\alpha_{ab}} - h^{-1}_{\alpha_{ab}})^i \left(h^{-1}_{e_c}\left\{h_{e_c}, V\right\}\right)^j \tag{3.40}$$

and

$$T[N] \propto \lim_{\Delta \to v(\Delta)} \sum_{\Delta \in \Sigma} N(v(\Delta)) \varepsilon^{abc} \varepsilon_{ijk} \left(h_{e_a}^{-1} \left\{ h_{e_a}, \left\{ C^E[1], V \right\} \right\} \right)^i \times \left(h_{e_b}^{-1} \left\{ h_{e_b}, \left\{ C^E[1], V \right\} \right\} \right)^j \left(h_{e_c}^{-1} \left\{ h_{e_c}, V \right\} \right)^k \quad (3.41)$$

respectively, where the limit $\Delta \to v(\Delta)$ is the same as the limit $\epsilon \to 0$. Note that the coordinate size of Δ, ϵ^3, does not appear in the expression since the contribution of the integral offsets the contribution of writing A_a^i and F_{ab}^i in terms of holonomies.

Now, one can quantise the Hamiltonian by promoting the variables in (3.40) and (3.41) to the corresponding quantum operators and turn the Poisson brackets into commutators (divided by $i\hbar$) between the variables. For instance, the quantum operator corresponding to $C^E[N]$ is

$$\hat{C}^E[N] \propto \frac{1}{i\hbar} \lim_{\Delta \to v(\Delta)} \sum_{\Delta \in \Sigma} N(v(\Delta)) \varepsilon^{abc} \delta_{ij} (\widehat{h_{\alpha_{ab}}} - \widehat{h_{\alpha_{ab}}^{-1}})^i \left(\widehat{h_{e_c}^{-1}} \left[\widehat{h_{e_c}}, \hat{V} \right] \right)^j . \quad (3.42)$$

Due to the presence of the volume operator, the Hamiltonian constraint only acts on the nodes of a state. Schematically, with an appropriate choice of position and orientation of the tetrahedra, the Hamiltonian constraint will act on a node of a spin network state $|S\rangle$ by (i) modifying the intertwiner associated with the node, (ii) creating new nodes and, (iii) modifying the spin j of two links connected to the node. A detailed explicit action of the Hamiltonian constraint on a simple node can be found in [6].

Physically relevant states $|\Psi\rangle \in \mathcal{H}_{phys}$ are those annihilated by the quantum Hamiltonian constraint $\hat{C}^{grav}[N]$

$$\hat{C}^{grav}[N] \, |\Psi\rangle = 0 \quad (3.43)$$

for any lapse function N. As we have mentioned at the beginning of Section 2, general relativity is a theory with no external time evolution, and the feature is inherited by its quantum version. This is reflected in (3.43). If there is external time evolution, we would instead have $\hat{C}^{grav}[N] \, |\Psi\rangle \propto \mathrm{d} \, |\Psi\rangle \, / \mathrm{d}t$ where t is the parameter labelling the external time. Since this is not the case, then each $|\Psi\rangle \in \mathcal{H}_{phys}$ is a solution describing a universe for the whole of spacetime.

For example, if $|\Psi_{Schw}\rangle$ corresponds to the Schwarzschild metric and $|\Psi_{FLRW}\rangle \neq |\Psi_{Schw}\rangle$ corresponds to the FLRW metric, then one will not evolve into the other. However, in the quantum version, we can have a universe described by the linear superposition of the two, which obviously satisfies (3.43). We end this section with a remark that in symmetry-reduced settings, it is possible to view the quantum Hamiltonian constraint equation as an evolution equation (see Section 4.6).

4 Loop Quantum Cosmology

4.1 Metric Variables

We will begin our discussion on loop quantum cosmology by reviewing basic results from standard cosmology following [25]. In a homogeneous and isotropic universe, spacetime is described by the FLRW metric

$$\mathrm{d}s^2 = -N(t)^2\mathrm{d}t^2 + a(t)^2 \left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2 \right) \right) \tag{4.1}$$

from which we can derive the corresponding Ricci scalar

$$R = 6 \left(\frac{\ddot{a}}{N^2 a} + \frac{\dot{a}}{N^2 a^2} - \frac{\dot{a}\dot{N}}{aN^3} + \frac{k}{a^3} \right). \tag{4.2}$$

Together with $\det g = -r^4 \sin^2\theta \, N(t)^2 a(t)^6/(1-kr^2)$, we obtain the reduced gravitational action

$$\begin{aligned} S^{iso}_{grav}[a, N] &= \frac{3\mathcal{V}}{8\pi G} \int \mathrm{d}t N a^3 \left(\frac{\ddot{a}}{N^2 a} + \frac{\dot{a}^2}{N^2 a^2} - \frac{\dot{a}\dot{N}}{aN^3} + \frac{k}{a^2} \right) \\ &= -\frac{3\mathcal{V}}{8\pi G} \int \mathrm{d}t \left(\frac{a\dot{a}^2}{N} - kaN \right) \end{aligned} \tag{4.3}$$

where we have ignored the constant boundary term that comes from integrating by parts. Due to homogeneity, the action (4.3) actually diverges. However, for the same reason, it is sufficient to consider any compact integration region. This is obvious from the appearance of $\mathcal{V} \equiv \int_{\mathcal{R}} \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi r^2 \sin^2\theta/\sqrt{1-kr^2}$ above. From the expression of $S^{iso}_{grav}[a, N]$, we note that the only canonical variable is a while N is only a Lagrange multiplier.

When we include matter (which must respect homogeneity and isotropy as well) with Hamiltonian H^{iso}_{matter}, further analysis would yield the Friedmann equation

$$\left(\frac{\dot{a}}{Na} \right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \frac{1}{a^3 \mathcal{V}} \frac{\partial H^{iso}_{matter}}{\partial N} \tag{4.4}$$

and the Raychaudhuri equation

$$\frac{(\dot{a}/N)^{\cdot}}{aN} = -\frac{4\pi G}{3}\left(\frac{1}{a^3\mathcal{V}}\frac{\partial H^{iso}_{matter}}{\partial N} - \frac{1}{Na^2\mathcal{V}}\frac{\partial H^{iso}_{matter}}{\partial a}\right) \tag{4.5}$$

In a homogeneous and isotropic universe, any matter is completely specified by two parameters, namely energy density ρ and pressure P which are defined as

$$\rho \equiv \frac{1}{a^3\mathcal{V}}\frac{\partial H^{iso}_{matter}}{\partial N}, \quad P \equiv -\frac{1}{3Na^2\mathcal{V}}\frac{\partial H^{iso}_{matter}}{\partial a}. \tag{4.6}$$

With these, we can put equations (4.4) and (4.5), respectively, into the more familiar form

$$\left(\frac{\dot{a}}{Na}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \tag{4.7}$$

$$\frac{(\dot{a}/N)^{\cdot}}{aN} = -\frac{4\pi G}{3}\left(\rho + 3P\right). \tag{4.8}$$

An important thing to note from the brief review above is that a can take the value zero. From equation (4.2) and (4.6) above, the Ricci scalar (which encodes the curvature of spacetime) and energy density diverges. Both evolution equations (4.7) and (4.8) are also useless at $a = 0$. Therefore, at this point, which is defined as the Big Bang, physics itself breaks down .

4.2 Connection Variables

For simplicity, the discussion will focus only on loop quantisation of a spatially flat $k = 0$ universe. As before, any spatial integration would inevitably diverge. Therefore, one has to restrict the analysis to a finite cell $\mathcal{C}$ with coordinate volume $\mathcal{V}$. After fixing the gauge, the connection and densitised triads can be assigned the form [16, 42]

$$A^i_a = c\mathcal{V}^{-1/3}\delta^i_a, \quad E^a_i = p\mathcal{V}^{-2/3}\delta^a_i \tag{4.9}$$

where both c and p are spatially constant. The latter two are the only coordinate-independent, gauge-invariant dynamical variables that satisfy the Poisson bracket

$$\{c, p\} = \frac{8\pi\gamma G}{3}. \tag{4.10}$$

However, they are sensitive to rescaling of the integration region; that is, $\mathcal{C} \to \lambda\mathcal{C}$ where $\lambda > 0$. A detailed discussion on rescaling can be found in [16]. These variables are related to the metric variables in Section 4.1 via

$$c = \gamma \dot{a} \mathcal{V}^{1/3}, \quad |p| = a^2 \mathcal{V}^{2/3}. \tag{4.11}$$

Since the Gauss and diffeomorphism constraints were automatically satisfied when the gauges are fixed to put A_a^i and E_j^b in the form of (4.9), the only non-trivial constraint left is the Hamiltonian constraint (2.16). Adding in matter Hamiltonian, H_{matter}, it now reads [42]

$$C^{iso} = -\frac{3}{8\pi G\gamma^2} c^2 \sqrt{|p|} + H_{matter}^{iso} \approx 0 \tag{4.12}$$

4.3 Kinematical Hilbert Space

To proceed analogously to the full theory, we need to find the holonomies of A_a^i along curves e and fluxes of E_j^b across surfaces $\mathcal{S}$. Thanks to the symmetry of the theory, we need not consider all curves and surfaces. It suffices to consider only straight lines and square surfaces.

For the holonomies, we can consider straight lines in any direction. For the sake of clarity, consider a straight line e_k of coordinate length l_k in the k^{th} coordinate. Then, the holonomy will be

$$\begin{aligned} h_{e_k}(c) &= \mathcal{P} \exp \int_{e_k} c\mathcal{V}^{-1/3} \delta_k^i \tau_i \mathrm{d}x^k \\ &= \exp\left(c\mathcal{V}^{-1/3}\tau_k l_k\right) \\ &= \cos\left(\frac{1}{2}\mu c\right)\mathbb{1} + 2\sin\left(\frac{1}{2}\mu c\right)\tau_k \end{aligned} \tag{4.13}$$

where there is no sum on the index k and we have defined a coordinate-independent parameter $\mu \equiv l_k \mathcal{V}^{-1/3}$ that labels the line. Here, note that the coordinate-independent, gauge-invariant information about the holonomy is captured in the pair

$(\cos(\mu c/2), \sin(\mu c/2))$. Therefore, it is sufficient to consider

$$\begin{aligned} \mathcal{N}_\mu(c) &\equiv \exp\left(\frac{i\mu c}{2}\right) \\ &= \cos\left(\frac{1}{2}\mu c\right) + i\sin\left(\frac{1}{2}\mu c\right) \end{aligned} \tag{4.14}$$

as the elementary functions of the configuration variable c. This introduces an Abelian artefact into the theory, which may not capture the structure arising from the fact that the connection A_a^i in the full theory is non-Abelian. This is analysed in [43] which also introduces a more careful way to quantise homogeneous models.

Similarly, for the flux, we can consider any square surface $\mathcal{S}$ to obtain

$$\begin{aligned} F_{\mathcal{S}}^{(f)}[p] &= \int_{\mathcal{S}} \mathrm{d}^2 y n_a p \mathcal{V}^{-2/3} \delta_i^a f^i \\ &= p\frac{A_{\mathcal{S},f}}{\mathcal{V}^{2/3}} \end{aligned} \tag{4.15}$$

where $A_{\mathcal{S},f}$ is the area of the surface times some orientation factor dependent on the choice of n_a and f^i. We can define $\sigma = A_{\mathcal{S},f}/\mathcal{V}^{-2/3}$ to label the surface and take $F_{\mathcal{S}}^{(f)}[p]$ as a whole to the other elementary variable conjugate to (4.14). However, σ does not carry any dynamical information and will simply be a linear multiplicative factor to $\hat{p}$ in the quantum version. Therefore, we can ignore it and take p alone as the other elementary variable [16]. These variables satisfy the Poisson bracket

$$\{\mathcal{N}_\mu(c), p\} = \frac{4\pi i\gamma\mu G}{3}\mathcal{N}_\mu(c). \tag{4.16}$$

With the elementary variables and Poisson bracket between them available, we can now construct the corresponding kinematical Hilbert space, $\mathcal{H}_{grav}^{iso}$. As in Section 3.4, the basic variables $\mathcal{N}_\mu(c)$ and p are promoted as operators on $\mathcal{H}_{grav}^{iso}$ such that their Poisson bracket (4.16) is promoted to a commutator

$$[\hat{\mathcal{N}}_\mu(c), \hat{p}] = -\frac{4\pi\gamma\mu G\hbar}{3}\hat{\mathcal{N}}_\mu(c). \tag{4.17}$$

Following the lead of the full theory in Section 3.1 and 3.3, one would expect to start by considering all possible straight lines and representations of the U(1)-holonomies (4.14). However, the Abelian artefact mentioned above hides the complexity and simplifies the situation.

Consider first what we will find if we indeed follow the full theory. Then, we would start by considering graph $\Gamma = (\mu_1, ...\mu_L)$ consisting of a collection of straight lines labelled with μ_l. Then, the basis states analogous to (3.14) will be

$$\begin{aligned} \langle c|\Gamma, j_L\rangle &\equiv \langle c|\Gamma, j_{\mu_1}, ..., j_{\mu_L}\rangle \\ &= \exp\left(i\frac{j_1\mu_1 c}{2}\right)...\exp\left(i\frac{j_L\mu_L c}{2}\right) \\ &= \exp\left(i\frac{(j_1\mu_1 + ... + j_L\mu_L)c}{2}\right) \end{aligned} \tag{4.18}$$

where we can combine the summation $\sum_l^L j_l\mu_l$ into a single parameter μ. Therefore, the basis for the Hilbert space $\mathcal{H}^{iso}_{grav}$ can be chosen to be

$$\langle c|\mu\rangle = \exp\frac{i\mu c}{2}. \tag{4.19}$$

Since this basis is gauge-invariant and the diffeomorphism constraint has already been satisfied, it is analogous to the s-knot states. In this way, however, μ_l's and j_l's that label the curves in a graph and the spin of the curve collapse to a single parameter μ. The different roles played by each type of label, then, are no longer obvious [42, 43].

Using the construction above, we actually obtain that the kinematical Hilbert space $\mathcal{H}^{iso}_{grav}$ is given by $L^2(\bar{\mathbb{R}}_{\text{Bohr}}, \mathrm{d}\mu_0)$ where $\bar{\mathbb{R}}_{\text{Bohr}}$ is the Bohr compactification of the real line $\mathbb{R}$ and $\mathrm{d}\mu_0$ is the Haar measure in this space [44]. General quantum states are given by the almost-periodic functions

$$\langle c|\Psi\rangle = \Psi(c) = \sum_n^N \Psi_n \exp\left(\frac{i\mu_n c}{2}\right) \tag{4.20}$$

where N is some finite positive integer, $\mu_n \in \mathbb{R}$ and $\Psi_n \in \mathbb{C}$ with inner product [17, 25]

$$\langle\Psi_1|\Psi_2\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathrm{d}c\,\overline{\Psi}_1\Psi_2. \tag{4.21}$$

The basis (4.19) satifies

$$\langle\mu|\mu'\rangle = \delta_{\mu\mu'} \tag{4.22}$$

which implies that the representation is not continuous in μ. Therefore, we cannot introduce the operator $\hat{c}$, for example, by taking the derivative of $\hat{\mathcal{N}}_\mu(c)$ with respect to μ [15, 25, 42].

In the connection representation, $\hat{\mathcal{N}_\delta}(c)$ acts multiplicatively on a basis state $\exp(i\mu c/2)$

$$\hat{\mathcal{N}_\delta}(c)e^{(\frac{i\mu c}{2})} = \mathcal{N}_\delta(c)e^{(\frac{i\mu c}{2})} = e^{(\frac{i(\mu+\delta)c}{2})}, \tag{4.23}$$

while $\hat{p}$ acts as a derivative operator

$$\hat{p}e^{(\frac{i\mu c}{2})} = -i\frac{8\pi\gamma l_P^2}{3}\frac{\mathrm{d}}{\mathrm{d}c}e^{(\frac{i\mu c}{2})} = \frac{4\pi\gamma l_P^2}{3}\mu e^{(\frac{i\mu c}{2})}. \tag{4.24}$$

In terms of abstract ket notation, it follows that

$$\hat{\mathcal{N}_\delta}(c)\,|\mu\rangle = |\mu+\delta\rangle\,,\ \ \hat{p}\,|\mu\rangle = \frac{4\pi\gamma l_P^2}{3}\mu\,|\mu\rangle \tag{4.25}$$

which shows that $|\mu\rangle$ are eigenstates of $\hat{p}$ with eigenvalues proportional to $\mu \in \mathbb{R}$. Note that, although μ is apparently continuous from (4.25), the spectrum is considered discrete in the sense that the eigenstates $|\mu\rangle$ are normalisable [16, 24, 25, 42]. We remark that since classically we have $|p| \propto a^2$, $|0\rangle$ corresponds to the state of the universe (or more precisely, the cell $\mathcal{C}$) at the classical Big Bang.

4.4 Volume Spectrum

The spectrum of the volume operator takes a simpler form than that of the full theory and can be evaluated explicitly. In terms of the metric variables, classically the physical volume of a region $\mathcal{R}$ is given by $V_R^{iso} = a^3\mathcal{V}_R$. Using (4.11), we obtain that the physical volume of our cell $\mathcal{C}$ is given by $V_\mathcal{C}^{iso} = |p|^{3/2}$. Then, (4.25) implies that the corresponding quantum volume operator $\hat{V}_{\mathcal{C}}^{iso}$ has a spectrum given by

$$\hat{V}_{\mathcal{C}}^{iso}|\mu\rangle = \left(\frac{4\pi\gamma}{3}\right)^{\frac{3}{2}} l_P^3 |\mu|^{\frac{3}{2}} |\mu\rangle \equiv V_\mu |\mu\rangle , \tag{4.26}$$

which is discrete. So, μ is directly related to the physical volume of the universe and large μ indicates a classical limit. The Big Bang state, however, still has zero volume.

4.5 Inverse Scale Factor

The main problem in cosmology as described by the FLRW metric (4.1) is the Big Bang singularity, that is, when the scale factor a vanishes. At this point, any function proportional to a^{-s} with $s > 0$ diverges. This problem will be cured if the spectrum of the quantum operator corresponding to $|p|^{-1/2}$ is bounded from above. However, the operator $\hat{p}$ has a zero eigenvalue (c.f. (4.25)) and, thus, is not invertible. So, in loop quantum cosmology, one instead analyses an operator that corresponds to the inverse of p in the classical limit. This is reasonable since $p^{-1/2}$ is not well-defined at the singularity, so that the operator corresponding to it in the classical limit may eventually be modified close to the singularity.

We start by introducing the classical identity [16, 42, 45]

$$\mathrm{sgn}(p)|p|^{r/2-1} = \frac{1}{2\pi\gamma G\delta r}\mathrm{Tr}\left(\sum_i^3 \tau_i e^{\delta c \tau_i}\left\{e^{-\delta c\tau_i}, V^{r/3}\right\}\right) \tag{4.27}$$

where $\mathrm{sgn}(p)$ is the sign function and $V \equiv V^{iso}$. Analogous to Section 3.6, this identity is then promoted to be an operator with the Poisson bracket turned into a commutator divided by $i\hbar$. Using $\exp(A\tau_i) = \mathbb{1}\cos(\frac{1}{2}A) + 2\tau_i \sin(\frac{1}{2}A)$ to calculate the trace, will obtain (for $\delta = r = 1$)

$$\begin{aligned}
\widehat{\mathrm{sgn}(p)|p|^{-1/2}} &= -\frac{i}{2\pi\gamma l_P^2}\mathrm{Tr}\left(\sum_i^3 \tau_i \widehat{e^{c\tau_i}}\left[\widehat{e^{-c\tau_i}}, \hat{V}^{1/3}\right]\right)\\
&= \frac{3}{4\pi\gamma l_P^2}(-2i)\left(\widehat{\sin\frac{c}{2}}\hat{V}^{1/3}\widehat{\cos\frac{c}{2}} - \widehat{\cos\frac{c}{2}}\hat{V}^{1/3}\widehat{\sin\frac{c}{2}}\right)\\
&= \frac{3}{4\pi\gamma l_P^2}\left(\widehat{e^{ic/2}}\left[\widehat{e^{-ic/2}}, \widehat{|p|}^{1/2}\right] - \widehat{e^{-ic/2}}\left[\widehat{e^{ic/2}}, \widehat{|p|}^{1/2}\right]\right)
\end{aligned} \tag{4.28}$$

which has the same eigenbasis $|\mu\rangle$ as $\hat{p}$ but with eigenvalues

$$\left(\widehat{\text{sgn}(p)|p|^{-1/2}}\right)_\mu |\mu\rangle = \left(\frac{3}{4\pi\gamma l_P^2}\right)^{1/2} \left(|\mu+1|^{1/2} - |\mu-1|^{1/2}\right)|\mu\rangle\,. \tag{4.29}$$

Form the spectrum above, we note that this operator has a vanishing eigenvalue at $\mu = 0$. This is in contrast with the classical value, which is divergent. At large $|\mu| \gg 1$, on the other hand, the operator does approach the inverse of $\hat{p}$. More importantly, the spectrum is bounded from above with an upper bound at $\mu = 1$

$$\left(\widehat{\text{sgn}(p)|p|^{-1/2}}\right)_1^{max} |1\rangle = \left(\frac{3}{2\pi\gamma l_P^2}\right)^{1/2} |1\rangle\,. \tag{4.30}$$

The occurrence of an upper bound at $\mu = 1$ instead of $\mu = 0$ is interesting. Consider defining the forward flow of time with an increasing value of μ as is the standard in cosmology (i.e. increasing time corresponds to increasing scale factor or volume). Then, if we reverse the time, there will be a divergence before one reaches the Big Bang ($\mu = 0$) if one remains classical, that is, $\hbar \to 0$. This heuristic evolution, of course, employs classical intuition of continuous flow of time. In the next section, we will review the proper evolution in loop quantum cosmology.

4.6 Quantum Evolution

To study the dynamics of this quantum universe, we need to quantise the Hamiltonian constraint (4.12). However, the classical expression contains the variable c^2 which, as noted above for the operator $\hat{c}$, has no direct quantum counterpart in loop quantum cosmology. As for the inverse of $\hat{p}$ in Section 4.5, the idea is to substitute it with an operator that reduces to c^2 in low curvature regime ($c \ll 1$) [46]. There is, however, no unique way to do so. An example would be to replace c^2 with $\delta^{-2}\sin^2\delta c$ [16, 46]. The action of the corresponding quantum version is

$$\begin{aligned}\frac{1}{\delta^2}\widehat{\sin\delta c}^{\,2}|\mu\rangle &= -\frac{1}{4\delta^2}\left(\widehat{e^{i\delta c}} - \widehat{e^{-i\delta c}}\right)^2|\mu\rangle \\ &= -\frac{1}{4\delta^2}\left(|\mu+4\delta\rangle - 2|\mu\rangle + |\mu-4\delta\rangle\right). \end{aligned} \tag{4.31}$$

For $\sqrt{|p|}$, one can quantise it directly or use the identity (4.27) to write it in terms of holonomies and volume first before quantising. Either approach will retain crucial properties of the quantum Hamiltonian constraint [16]. If one chooses the latter, which is closer to the full theory, then it reads

$$\hat{C}^{iso} = -\frac{3}{32\pi^2\gamma^3 G l_P^2 \delta^3} \widehat{\sin\delta c}^2 \operatorname{sgn}(\mu) \left(\widehat{e^{-\frac{1}{2}i\delta c}} \hat{V} \widehat{e^{\frac{1}{2}i\delta c}} - \widehat{e^{\frac{1}{2}i\delta c}} \hat{V} \widehat{e^{-\frac{1}{2}i\delta c}} \right) + \hat{H}^{iso}_{matter} \tag{4.32}$$

where $\operatorname{sgn}(p)$ is replaced by $\operatorname{sgn}(\mu)$ by virtue of (4.25). Acting this form of Hamiltonian constraint on a volume eigenstate $|\mu\rangle$, we obtain

$$\begin{aligned} \hat{C}^{iso}|\mu\rangle &= \frac{3}{128\pi^2\gamma^3 G l_P^2 \delta^3} \operatorname{sgn}(\mu) \left(V_{\mu+\delta} - V_{\mu-\delta}\right) \left(|\mu+4\delta\rangle - 2|\mu\rangle + |\mu-4\delta\rangle\right) \\ &\quad + \hat{H}^{iso}_{matter}|\mu\rangle . \end{aligned} \tag{4.33}$$

Similar to the full theory, for a general quantum state $|\Psi\rangle$ to be an element of $\mathcal{H}^{iso}_{phys}$, it must be annihilated by (4.32), that is,

$$\hat{C}^{iso}|\Psi\rangle = 0. \tag{4.34}$$

Note that $|\Psi\rangle$ also contains information about all matter content in the theory. From (4.33), volume eigenstates obviously do not satisfy this criteria. As we remark at the end of Section 3, it is possible to view (4.34) as an evolution equation. For that, we need to introduce the notion of internal time.

The idea is fairly simple: choose a particular (set of) degree of freedom as the internal time and view the configurations of other degrees of freedom with respect to them. Although the concept is not very familiar in physics, or rather it is not made explicit, we actually use it in everyday life. For instance, when one carries out an experiment at 7.01 am, what it really means is that he/she does so when his/her watch shows 7.01 am.

In our case here, we can choose the volume of the universe as the internal time and expand a general physical quantum state $|\Psi\rangle$ in terms of volume eigenstates [16, 47]

$$|\Psi\rangle = \sum_{\mu} \Psi_{\mu} \, |\mu\rangle \tag{4.35}$$

where the sum runs over all possible values of μ. In this way, we can view Ψ_{μ} as the wave function of the universe at time μ. This is analogous to specifying the configurations of matter fields with respect to values of a in classical cosmology.

Due to the orthonormality condition (4.22) of volume eigenstates $|\mu\rangle$, the constraint equation (4.34) implies a difference equation [16]

$$\begin{aligned} &\mathrm{sgn}(\mu + 4\delta) \left(V_{\mu+5\delta} - V_{\mu+3\delta}\right) \Psi_{\mu+4\delta} - 2\mathrm{sgn}(\mu) \left(V_{\mu+\delta} - V_{\mu-\delta}\right) \Psi_{\mu} \\ &+ \mathrm{sgn}(\mu - 4\delta) \left(V_{\mu-3\delta} - V_{\mu-5\delta}\right) \Psi_{\mu-4\delta} = \frac{128\pi^2\gamma^3 G l_P^2 \delta^3}{3} \hat{H}^{iso}_{matter}(\mu)\Psi_{\mu}. \end{aligned} \tag{4.36}$$

The first thing to note is that we now have a discrete time evolution with a constant time-step of 4δ instead of a continuous one.[2] This is a direct effect of spacetime quantisation. Next, the evolution equation (4.36) is well-defined at $\mu = 0$, provided that $\hat{H}^{iso}_{matter}(\mu)$ is similarly well-defined. Therefore, one can evolve a given initial value of Ψ_{μ}, for instance $\mu > 0$, backwards and passes through or jumps over the $\mu = 0$ Big Bang [16, 47].

4.6.1 Big Bounce

The simplest matter field known is the free massless scalar φ. In this case, the matter Hamiltonian H^{iso}_{matter} takes the form

$$H^{iso}_{matter} = \frac{p_{\varphi}^2}{2|p|^{3/2}} \tag{4.37}$$

where p_{φ} is the canonical momentum conjugate to φ. The matter sector of the model can be quantised using standard Schrodinger representation, resulting in the total

[2] We note that one can also introduce $\delta(\mu)$ so that the time-step will no longer be constant. For further discussion on this, see [16].

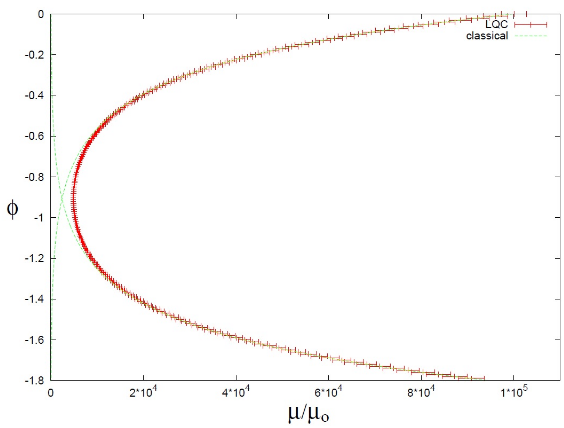

Figure 5: A result from [22] showing that the classical Big Bang is replaced with a quantum Big Bounce. In this diagram, μ_0 is a parameter analogous to δ in (4.27).

kinematical Hilbert space given by $\mathcal{H}_{kin}^{total} = L^2(\bar{\mathbb{R}}_{\text{Bohr}}, \mathrm{d}\mu_0) \otimes L^2(\mathbb{R}, \mathrm{d}\varphi)$ [22]. Upon quantisation, the variable $|p|^{-3/2}$ is substituted with the operator introduced in Section 4.5 while p_φ is promoted to a derivative operator in φ-representation

$$\hat{p}_\varphi \Psi(\mu, \varphi) = -i\hbar \frac{\partial}{\partial \varphi} \Psi(\mu, \varphi). \tag{4.38}$$

In this way, the constraint equation (4.34) can be rearranged to take the form

$$\begin{aligned} \frac{\partial^2}{\partial \varphi^2} \Psi(\mu, \varphi) &= \frac{2}{\hbar^2} B(\mu)^{-1} \hat{C}_{grav}^{iso} \Psi(\mu, \varphi) \\ &\equiv -\Theta \Psi(\mu, \varphi) \end{aligned} \tag{4.39}$$

where $B(\mu)$ is the eigenvalue of $\widehat{|p|^{-3/2}}$ and $\hat{C}_{grav}^{iso}$ is the quantum operator corresponding to the first term in (4.12). From the constraint equation above, it is very appropriate to assume φ as the internal time. Indeed, this is the choice made in [21, 22, 23] by Ashtekar, Pawlowski and Singh. Together with an alternative factor ordering in quantising C_{grav}^{iso}, they derived a general solution to the Hamiltonian constraint equation (4.39) and investigated the evolution of μ with respect to internal time φ using numerical methods. The result is the replacement of the classical Big Bang with a quantum Big Bounce, as shown in Figure 5.

4.6.2 Effective Equations

The model of the homogeneous isotropic universe with the free massless scalar field φ above has also been studied using effective equations, for instance, in [16, 46, 48] by Bojowald. In the analysis, one first rearranges the classical Hamiltonian constraint (4.12) to take the form

$$p_\varphi = \pm H(c, p). \tag{4.40}$$

Upon quantisation, this form of the constraint equation allows one to analyse the model as a system evolving over time φ and governed by the Hamiltonian $H(c, p)$. In particular, one can use equations familiar in quantum mechanics such as

$$\frac{\mathrm{d}}{\mathrm{d}\varphi} \langle \hat{O} \rangle = \frac{\langle [\hat{O}, \hat{H}] \rangle}{i\hbar} \tag{4.41}$$

where $\hat{O}$ is an operator defined for the system. The details of the analysis can be found in [16, 46, 48].

An advantage of this method is that we can recover equations of motion of the classical type, which nevertheless incorporate quantum effects as corrections to the equation. In the case of the homogeneous isotropic universe with the free massless scalar, we can derive a modified Friedmann equation [16]

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{\text{free}}\left(1 - \frac{\rho_{\text{free}}}{\rho_{\text{crit}}}\right) + O(\hbar) \tag{4.42}$$

where ρ_{free} is the energy density of the free massless scalar field φ and ρ_{crit} is a critical density around which correction terms become important.

Since we now have a classical equation, we can use the usual intuitive notion of time in cosmology. One can see that as we look at the reverse evolution of the universe, the right-hand side of the modified Friedmann equation (4.42) approaches zero (ignoring $O(\hbar)$ contributions) as the energy density ρ_{free} approaches ρ_{crit}. This implies the existence of an extremum in the evolution of the scale factor a. We can similarly derive a modified Raychaudhuri equation [16]

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_{\text{free}}\left(1 - \frac{3\rho_{\text{free}}}{2\rho_{\text{crit}}}\right) \tag{4.43}$$

from which we see that $\ddot{a} > 0$ as ρ_{free} approaches ρ_{crit}. This implies that the extremum above is a minimum point. Therefore, we again validate the Big Bounce picture as in Section 4.6.1.

5 Conclusion

Since its introduction to the theoretical physics community in 1987, loop quantum gravity has developed significantly, to the extent that it has evolved into one of the largest research focuses in the field of quantum theory of gravity [7, 11]. The most appealing feature of the theory is its lack of additional a priori assumptions about the nature of quantum spacetime. The inputs are only quantum mechanics and general relativity, which have been very successful within their regime of applicability. Its main result, namely the discreteness of the spectra of area and volume operators, gives important insights into the quantum nature of space on the Planck scale. Nevertheless, the theory is still far from complete.

As noted, one of the main open problems is the implementation of the Hamiltonian constraint. This is a crucial step to the recovery of a quantum spacetime picture of loop quantum gravity rather than “quantum space at fixed time”. The quantisation of the Hamiltonian constraint, as briefly reviewed in Section 3.6, still suffers from unsettled problems (see, for example, [6]). In the last few years, Thiemann has introduced another idea to confront this problem. Specifically, he proposed the combination of the smeared Hamiltonian constraints for all smearing functions into a single constraint known as the Master constraint [49].

Alternatively, there are attempts to perform loop quantisation of general relativity in a covariant way. This results in the spin-foam formalism [50]. In a sense, this formalism avoids the Hamiltonian constraint altogether. Another closely related approach to the study of the dynamics of loop quantum gravity is the group field theory approach [51]. Another important open problem in loop quantum gravity is to prove that it gives general relativity in the low-energy, or classical, limit [7].

Due to the underlying symmetry in loop quantum cosmology, it is more developed in these areas. As we have seen in Section 4.6, the Hamiltonian constraint equation can even be viewed as an evolution equation that is closer to our physical intuition on spacetime. There exists detailed analytical and numerical study of an isotropic, spatially flat universe with a free massless scalar field [21, 22, 23]. Also, an investigation of the semiclassical limit yields the correct classical behaviour [52].

However, the purpose of loop quantum cosmology is to provide a way to "test" the theory in a simpler setting. To that end, application to an isotropic, homogeneous model alone is insufficient. Indeed, efforts are being made to extend the study to anisotropic or/and inhomogeneous models of cosmology, for example in [15, 16].

Another important application of loop quantum gravity that is not discussed in this dissertation is the application to black holes. Analogous to the cosmological case, loop quantisation solves the singularity problem in the Schwarzschild black hole [53, 54]. There is also discussion of a paradigm to describe black hole evaporation in the context of loop quantum gravity [55]. In addition, the Bekenstein-Hawking entropy of black hole of surface area A has also been calculated (see, for example, [56, 57]).

We conclude that although there is still significant research to be done to finalise loop quantum gravity, the theory has also progressed steadily since its inception. More importantly, loop quantum gravity provides a framework to describe quantum mechanics in a background independent way.

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